A

SYNOPSIS

ON

"A STUDY OF ANALYTIC FUNCTIONS ASSOCIATED WITH POLYNOMIALS"

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Swami Ramanand Teerth Marathwada University

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By

Mr. Hake Dnyanoba Subhash

Research student

N.E.S. Science College, Nanded

Under the guidance of

Dr. N. S. Jadhav

Department of Mathematics

Madhavrao Patil College, Palam

Dist: Parbhani-431720.

1 Introduction

Geometric function theory (GFT) is an active subject of Mathematics which is a sub-branch of complex analysis. The Study of GFT involves finding the relationship between the analytical properties of the function of complex variable $f(z)$, and the geometrical properties of the image domain $D = f(E)$ where $E = \{z \in \mathbb{C} : |z| < 1\}$ 1ies the unit disc. We attempt to investigate some geometrical aspects of Univalent and Multivalent functions. We have obtained some nice results of coefficients estimates, convolution property, growth and distortion bounds, extreme points, criterion for univalency, p -valency and extremal functions for these classes. The study will continue the investigation of several properties of analytic and univalent functions and the different subclasses like S^*- starlike, C- convex, K -close-to-convex, uniformly starlike and convex, parabolic starlike and convex etc. Application of fractional derivate operator, fractional integral operator, and hyper geometric functions to different subclasses of polylogarithm and univalent and multivalent functions is also of interest to us. The study of differential subordinations and criteria for univalency and their applications to various subclasses of univalent as well as multivalent functions is also under consideration. Using GFT and utilizing the theory of complex analysis we will try to find the connections with other branches of mathematics and applications in science and engineering. In this venture we will define several subclasses of univalent and multivalent functions with multiple properties. Geometric functions theory is quite old and has become one of the most outstanding branches of complex analysis. This study involves finding the relationship between the analytical properties and the geometrical properties of the image domain where is the unit disc. Attempts to solve the numerous conjectures in geometric functions theory have resulted in enriching the

classical geometric function theory in several directions. Geometric function theory primarily comprises of Univalent and Multivalent functions and plays a central role in the development of complex analysis. We aim at investigating some geometrical aspects of Univalent and Multivalent functions. The theory of conformal mappings is intimately related with the theory of boundary value problems(b.v.p.) for harmonic functions. Hence GFT has many applications in mathematical physics. The need also arises for good numerical methods for construction of conformal maps. However, this interplay works only in two-dimensions. In three-dimensions due to the classical theory of Liouville, there are only few and trivial conformal mappings. In higher dimensions the powerful tool of conformal mapping fails. GFT has and continues to have a profound impact on other branches of Mathematics like: 1. Complex analysis 2. Harmonic analysis 3. Functional analysis 4. Algebra 5. Partial differential equations 6. Dynamics 7. Geometry 8. Topology 9. Global analysis.

There are many conjectures in Mathematics which have been solved by the use of Geometric functions theory, for instance the, Bieberbach Conjecture. This conjecture has been solved for some values of n and for all values of n for certain subclasses of univalent functions, but the full conjecture still remains open. The theory of regular functions (univalent functions) was studied by Koebe [18] in 1907 . He first studied the classical Riemann Mapping Theorem; existence theorems for canonical conformal mapping. The theory of regular functions (univalent function) is one of most interesting subject in G.F.T. Gronwall [12] studied and proved the area theorem in 1914 and Bieberbach has estimated in 1916 the second coefficient of univalent function under normalized conditions. In 1984 Louis de Branges [5] solved the Bieberbach conjecture that the famous coefficient problem , in that he

suggested various directions and approaches about geometric function theory . From the last 100 years geometric functions theory have developed vastly.

2 Objectives

We propose to study in the following areas:

(i). To solve Bessels equations by using convolution properties.

(ii). To solve differential operators by using subordination principles.

(iii). The study of q- generalized bessel functions of differential and integral operators and find out their possible new applications.

(iv). To study of q-analogues on various generalization of coefficient bounds of meromorphic Bi-univalent differential and integral operators.

(v). To obtain partial sums and neighborhood properties of subclasses.

(vi). To find coefficients bounds of bi-univalent and multivalent functions.

(vii). To obtain distortion bounds for the functions of these classes.

(viii). To obtain certain inclusion relations such as convex linear combinations and their applications.

(ix). To examine the applications of these subclasses.

(x). To find various applications of special polynomials like Gegangauer, Lucas, Pascal, Borel, Binomial distribution series.

(xi). To study of q-analogues on various generalization of differential and integral operators of Bi-univalent functions.

(xii). To define new class and find extreme points, redius of starlikeness, derive Fekete-Szego type coefficient inequalities and its application with Lucas polynomial by expressing it in convolution form.

(xiii). To study the inclusion property and other geometric properties by introducing some new classes of analytic functions defined using Kornatu Integral.

(xiv). To discuss geometric properties by defining some new classes of analytic function using subordination and differential subordination.

(xv). To obtain sufficient conditions of starlikeness, convexity and close-to-convexity, alpha-convexity, spiralikeness for the newly defined classes of analytic functions by using the principles of differential subordination.

3 study the various methods of univalent function

The study of univalent is one of the main branches of geometric function theory and one of the most fascinating aspects of the theory of analytical functions of a complex variable. Koebe [18] in 1907 in his investigation, Gronwalls [12] proof of the area theorem in 1914 and Bieberbachs [3] assessment for the second coefficient of a normalized univalent function in 1916, showed the finesse of the theory of univalent function. Therefore number of mathematicians were attracted by those interesting problems and motivated to develop various new methods for complex analysis.

The basic methods of the theory of univalent functions and some of the results obtained from them are given below.

i. The method of integral representations.

ii. The method of boundary integration.

iii. The area method.

iv. Lowner's method.

v. Variational methods.

vi. The method of the extremal metric.

vii. The method of symmetrization.

4 Importance of Study

It is expected that most of the results which will obtain in this survey may find applications in the solution of certain differential and integral equations arising problems of physical science, engineering and statistics.We have discuss basic properties,convolution properties, partial sums, weighted mean, arithmetic mean, linear combination, inclusion relation and neighborhood properties, application of various polynomials and binomial,Pascal,Touchard and other properties for the subclasses.

By making use of the properties of convolution and theory of subordination, several subclasses of meromorphic functions are introduced. Subjecting each convoluted derived function in the class to be subordinated to a given normalized convex function with positive real part, these subclasses extend the classical subclasses of meromorphic starlikeness, convexity, close-to-convexity, and quasi-convexity. Class relations, as well as inclusion and convolution properties of these subclasses are investigated.

5 Review of Historical Background

A Fractional calculus has its origin in the question of the extension of meaning. A well known example is the extension of meaning of real numbers to

complex numbers, and another is the extension of meaning of factorials of integers to factorials of complex numbers.

Koebe [18] pioneered the study of univalent functions in 1907, when he discovered that the range of all functions in S contains a common disc $|w| < b$, where b is an absolute constant. The Koebe function demonstrates that $b \leq 1/4$. Bieberbach [3] established that b can be taken to be $1/4$, and he also demonstrated that if S, then equality occurs only for rotations of the Koebe function. Because the Koebe function appears to be an external function for so many problems in class S, Bieberbach [3] suspected that it maximises for every n, resulting in the famous Bieberbach conjecture, first proposed in 1916.

For many years, this famous problem served as a challenge, inspiring the development of important concepts and ingenious techniques that now serve as the foundation of the entire subject. Bernardi [1], Hayman [13], and Goodman [9] have reviewed various developments in the theory of univalent functions. Certain aspects of the subject have also been addressed in books by Nehari [19], Duren [7], Goluzin [8], and Jenkins [14]. Bernardi's [2] exhaustive Bibliography of schlicht functions contains over 4,200 references available in the literature on the theory of univalent functions up to 1981. Rusheweyh's book [25] provides an overview of all the results in convolution theory for univalent functions. Robertson [21] introduces the concept of order of Starlikeness as well as the concept of convex functions of order. Still another subclass of Starlike functions that has occupied wide attention of workers in the field in recent years is the class of strongly Starlike functions introduced by Brannan and Kirwan [4] and Stankiewicz [27].

6 Basic concepts

Let A denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions and satisfy the following usual normalization condition $f(0) = f'(0) - 1 = 0$. We denote by S the subclass of A consisting of functions $f(z)$ which are all univalent in E. A function $f \in A$ is a starlike function of the order $\alpha, 0 \leq \alpha < 1$, if it satisfy

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, (z \in E).$$

We denote this class with $S^*(\alpha)$.

A function $f \in A$ is a convex function of the order $\alpha, 0 \leq \alpha < 1$, if it satisfy

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, (z \in E).$$

We denote this class with $K(\alpha)$.

Let T denote the class of functions analytic in E that are of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0 \ (z \in E)$$

and let $T^*(\alpha) = T \cap S^*(\alpha),\ C(\alpha) = T \cap K(\alpha)$. The class $T^*(\alpha)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [26] and others.

A function $f \in A$ is said to be in the class of uniformly convex functions

of order γ and type β, denoted by $UCV(\beta,\gamma)$, if

$$Re\left\{1+\frac{zf''(z)}{f'(z)}-\gamma\right\}>\beta\left|\frac{zf''(z)}{f'(z)}\right|,$$

where $\beta\geq 0,\gamma\in[-1,1)$ and $\beta+\gamma\geq 0$ and it is said to be in the class corresponding class denoted by $SP(\beta,\gamma)$, if

$$Re\left\{\frac{zf'(z)}{f(z)}-\gamma\right\}>\beta\left|\frac{zf'(z)}{f(z)}-1\right|,$$

where $\beta\geq 0,\gamma\in[-1,1)$ and $\beta+\gamma\geq 0$. Indeed it follows that

$$f\in UCV(\gamma,\beta)\Leftrightarrow zf'\in SP(\gamma,\beta).$$

For $\beta=0$, we get respectively, the classes $K(\gamma)$ and $S^*(\gamma)$. The function of the class $UCV(1,0)\equiv UCV$ are called uniformly convex functions, were introduced and studied by Goodman with geometric interpretation in [10,11] and uniformly starlike functions in [11]. The class $SP(1,0)\equiv SP$ is defined by Ronning [23]. The classes $UCV(1,\gamma)\equiv UCV(\gamma)$ and $SP(1,\gamma)\equiv SP(\gamma)$ are investigated by Ronning in [22]. For $\gamma=0$, the classes $UCV(\beta,0)\equiv\beta-UCV$ and $SP(\beta,0)\equiv\beta-SP$ are defined respectively, by Kanas et al. in [16] and [17].

Let Σ denote the class of the functions of the form

$$f(z)=\frac{1}{z}+\sum_{m=1}^{\infty}a_m z^m$$

which are regular in domain $E=\{z\in\mathbb{C}:0<|z|<1\}$ with a simple pole at the origin with residue 1 there. Let Σ_s and $\Sigma^*(\alpha)$ and $\Sigma_k(\alpha),0\leq\alpha<1$, denote the subclasses of Σ that are univalent, meromorphically starlike of order α and meromorphically convex of order α respectively. Analytically

$f(z)$ is in $\Sigma^*(\alpha)$ if and only if

$$Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha, \qquad z \in E.$$

Similarly, $f \in \Sigma_k(\alpha)$ if and only if $f(z)$ is in $\Sigma^*(\alpha)$ and satisfies

$$Re\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \alpha, \qquad z \in E.$$

It being understood that if $\alpha = 1$ then $f(z) = \frac{1}{z}$ is the only function which is $\Sigma^*(1)$ and $\Sigma_k(1)$. The classes $\Sigma^*(\alpha)$ and $\Sigma_k(\alpha)$ have been extensively studied by Pommerenke [20], Clunie [6], Royster [24] and Juneja and Reddy [15].

7 Year-wise Scheme of the Project

From the objectives of the research project, it is clear that the said study is very vast and therefore, it is difficult to chalk out the concrete time plan for its completion. However the rough sketch of the two-year year-wise work-plan of the study of the present research project is as follows.

In the first year, the basic literature concerning the fundamental background concepts of the research project will be collected and studied with their relevance on the chosen topic. This process also includes attendance of seminars and guest lectures on the mathematical analysis and applications. In the second year of proposed plan, we shall collect the literature concerning the fractional calculus and special functions along with some literature covering the applications in Geometric function Theory. This process includes the collection of research papers along with current work of reputed mathematicians for the detailed study from various angles of conclusions. In the third year discussion and presentation of the new ideas will be made with

other researches working in the held of undertaken research project along with the publication of the new findings in the reputed referred journals. This step also includes the document preparation of the research project for its submission to university for the award of Ph. D. degree in the subject of mathematics. The review of the research work done will be taken after every six month and will be presented before the concerned committee in the form of six monthly progress reports during the three-year tenure of the work.

8 Conclusion

It is claimed that the results of this thesis will be a new basic and original contribution to the subject of Geometric function Theory and applications. It is hoped that the present work will also be useful to scientists, engineers and mathematicians, who are engaged in analyzing the universal classical as well as anomalous phenomena governed by univalent functions differential and integral equations to predict and design the future strategy of the study in a better way.

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Hake Dnyanoba Subhash,

Research Scholar

At:Bhakarwadi post Ajandonda(bk)

Tq.Chakur Dist: Latur-413 513.

Email: dhake4050@gmail.com

Dr. N.S.Jadhav

Research Guide

Department of Mathematics,

Madhavrao Patil College Palam,

Dist. Parbhani-431 720

Email:nsjadhav11@gmail.com